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Critique of Komar's solution to the factor ordering problem of the constraint algebra of general relativity

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Abstract. It is shown that a solution to this problem proposed by Komar is invalid. We prove that the canonically quantised ADM constraints must be symmetrically ordered.

1. Introduction

We have recently completed an analysis that suggests that no consistent canonical quantisation of the constraint algebra of general relativity exists (Ryan 1982).

A resolution of the problem has been proposed using non-Hermitian constraints (Komar 1979a). Here I should like to point out a number of errors in this analysis.

In § 2 I give a brief outline of the ADM formulation. Section 3 describes the Dirac quantisation scheme and states the general problem. Section 4 explains a number of difficulties in Komar's 'solution'. In § 5 we prove a theorem which effectively shows that no solution along the lines suggested by Komar can be consistent. Finally § 6 contains some conclusions.

2. The ADM formulation of general relativity

The ADM (Arnowitt *et al* 1962) Hamiltonian formulation of the dynamics of general relativity uses as configuration space coordinates the metric induced on space-like hypersurfaces. These we denote by the symbols $g_{\bar{\mu}\bar{\nu}}$. Barred indices label spatial components. Conjugate to the configuration space coordinates ADM define the momenta:

$$p^{\bar{\mu}\bar{\nu}} := \frac{\delta I}{\delta(\partial_0 g_{\bar{\mu}\bar{\nu}})}. \tag{1}$$

I is, modulo a total divergence, the Hilbert action. These satisfy the (equal-time) fundamental Poisson bracket relations

$$\{g_{\bar{\mu}\bar{\nu}}(x), p^{\bar{\sigma}\bar{\tau}}(x')\} = \tfrac{1}{2}(\delta^{\bar{\sigma}}_{\bar{\mu}}\delta^{\bar{\tau}}_{\bar{\nu}} + \delta^{\bar{\tau}}_{\bar{\mu}}\delta^{\bar{\sigma}}_{\bar{\nu}})\delta(x, x'). \tag{2}$$

The invariance of the action I under local translations $x_\mu \to x_\mu + \xi_\mu$ leads to a set of four constraints $\mathscr{H}_\mu$.

Reflecting their geometric nature, these constraints verify a closed algebra with respect to the Poisson bracket.

$$\{\mathscr{H}_{\bar{\mu}}(x), \mathscr{H}_{\bar{\nu}}(x')\} = \mathscr{H}_{\bar{\mu}}(x)\delta_{,\nu}(x, x') + \mathscr{H}_{\bar{\nu}}(x')\delta_{,\bar{\mu}}(x, x') \tag{3}$$

$$\{\mathscr{H}_{\bar{\mu}}(x), \mathscr{H}_{\perp}(x')\} = \mathscr{H}_{\perp}(x)\delta_{,\bar{\mu}}(x, x') \tag{4}$$

$$\{\mathscr{H}_{\perp}(x), \mathscr{H}_{\perp}(x')\} = g^{\bar{\mu}\bar{\nu}}(x)\mathscr{H}_{\bar{\nu}}(x)\delta_{,\bar{\mu}}(x, x') + g^{\bar{\mu}\bar{\nu}}(x')\mathscr{H}_{\bar{\mu}}(x')\delta_{,\bar{\nu}}(x, x'). \tag{5}$$

Explicitly the constraints expressed in the ADM variables are

$$\begin{aligned}\mathscr{H}_{\bar{\mu}} &= -2p_{\bar{\mu}}{}^{\bar{\nu}}|_{\bar{\nu}} \\ &= -2g_{\bar{\mu}\bar{\sigma}}p^{\bar{\sigma}\bar{\nu}}{}_{,\bar{\nu}} - (2g_{\bar{\sigma}\bar{\mu},\bar{\nu}} - g_{\bar{\sigma}\bar{\nu},\mu})p^{\bar{\nu}\bar{\sigma}}\end{aligned} \tag{6}$$

$$\mathscr{H}_{\perp} = g^{-1/2}(p^{\bar{\mu}\bar{\nu}}p_{\bar{\mu}\bar{\nu}} - \tfrac{1}{2}p^2) - g^{1/2}R \tag{7}$$

where $|$ denotes the induced covariant derivative, the index $\perp$ labels components normal to the hypersurfaces, g is the determinant of the three metric. R is the scalar curvature of the hypersurface, and $p := g_{\bar{\mu}\bar{\nu}}p^{\bar{\mu}\bar{\nu}}$.

3. The Dirac quantisation scheme

Canonical quantisation is implemented by the following correspondence, where circumflexes denote quantum operators:

$$g_{\bar{\mu}\bar{\nu}} \to \hat{g}_{\bar{\mu}\bar{\nu}} \tag{8}$$

$$p^{\bar{\mu}\bar{\nu}} \to \hat{p}^{\bar{\mu}\bar{\nu}} \tag{9}$$

$$\{\ ,\ \} \to (\mathrm{i}\hbar)^{-1}[\ ,\ ] \tag{10}$$

$$\mathscr{H}_{\mu} = 0 \to \hat{\mathscr{H}}_{\mu}|\psi\rangle_{\text{phys}} = 0. \tag{11}$$

The square bracket denotes the quantum commutator. The quantum condition (11) asserts that vectors representing physical states are gauge (i.e. coordinate) invariant.

Consistency leads us, after Dirac, to require

$$[\hat{\mathscr{H}}_{\mu}, \hat{\mathscr{H}}_{\nu}]|\psi\rangle_{\text{phys}} = 0. \tag{12}$$

This should follow directly from (11), that is, without imposing further conditions on the $|\psi\rangle_{\text{phys}}$.

Consequently we anticipate that the quantum constraint algebra will be as follows:

$$[\hat{\mathscr{H}}_{\bar{\mu}}, \hat{\mathscr{H}}_{\bar{\nu}}] = \mathrm{i}\hbar(\hat{\mathscr{H}}_{\bar{\mu}}\delta_{,\bar{\nu}} + \hat{\mathscr{H}}_{\bar{\nu}}\delta_{,\bar{\mu}}) \tag{13}$$

$$[\hat{\mathscr{H}}_{\bar{\mu}}, \hat{\mathscr{H}}_{\perp}] = \mathrm{i}\hat{\mathscr{H}}_{\perp}\delta_{,\bar{\mu}} \tag{14}$$

$$[\hat{\mathscr{H}}_{\perp}, \hat{\mathscr{H}}_{\perp}] = \mathrm{i}(\hat{g}^{\bar{\mu}\bar{\nu}}\mathscr{H}_{\bar{\nu}}\delta_{,\bar{\mu}} + \hat{g}^{\bar{\mu}\bar{\nu}}\mathscr{H}_{\bar{\mu}}\delta_{,\bar{\nu}}). \tag{15}$$

The crucial point to note is that the $\hat{g}^{\bar{\mu}\bar{\nu}}$ on the RHS of (15) are ordered to the left of the $\hat{\mathscr{H}}_{\bar{\mu}}$.

It is hoped to find an ordering (or orderings) of the $\hat{g}$ and $\hat{p}$ in the $\hat{\mathscr{H}}_{\mu}$ such that, subject to the CCR's, the algebra (13)–(15) is verified. This problem seems to have been first investigated by Anderson (1963).

4. Critique of Komar's analysis

Komar's contention is that the non-symmetric ordering of the 'Hamiltonian' (normal) constraint

$$\hat{\mathscr{H}}_{\perp}^{K} = \hat{g}^{-3/2}\hat{p}^{\bar{\mu}\bar{\nu}}\hat{g}(\hat{g}_{\bar{\mu}\bar{\sigma}}\hat{g}_{\bar{\nu}\bar{\tau}} - \tfrac{1}{2}\hat{g}_{\bar{\mu}\bar{\nu}}\hat{g}_{\bar{\sigma}\bar{\tau}})p^{\bar{\sigma}\bar{\tau}} - \hat{g}^{1/2}\hat{R} \tag{16}$$

along with a symmetric ordering of the hypersurface translation constraints $\hat{\mathscr{H}}_{\bar{\mu}}$, solves the problem.

Komar (1976b) presents an argument that claims to demonstrate that constraints that generate dynamics must be non-Hermitian. This is intended to justify the non-symmetric ordering (16). The argument is, however, fallacious. Firstly, it must be remarked that Hermiticity is meaningless until an inner product has been defined on the Hilbert space.

The argument would appear to rest on the assumption that

$$\hat{K}|\psi\rangle_{\text{phys}} = 0 \Rightarrow {}_{\text{phys}}\langle\psi|\hat{K}^{+} = 0. \tag{17}$$

As is made clear by Dirac (1962) this is invalid. Almond (1980) has provided an alternative refutation from a somewhat different point of view.

It is not immediately clear, since the constraints are non-observable, that taking them to be non-Hermitian is inadmissible. However, it can be shown that for consistency they must be 'normal', that is, they must weakly commute with their adjoints. This can be seen by noting that, if $\hat{C}_a$ annihilates physical state vectors, then $\hat{C}_b^{+}$ generates symmetry transformations. Requiring such transformations to preserve the condition

$$\hat{C}_a|\psi\rangle_{\text{phys}} = 0 \tag{18}$$

leads directly to

$$[\hat{C}_b^{+}, \hat{C}_a] \approx 0. \tag{19}$$

(This is a rather non-standard use of the term normal.)

A short calculation shows that Komar's ordering is not normal.

5. Quantum ordering of the constraints

For the translational constraints of general relativity, the following theorem can be proved. It is assumed that the representation is such that $\hat{g}$ and $\hat{p}$ are given by Hermitian operators. In view of the Stone–von Neumann theorem (Emch 1972) that states that all operator representations of the CCR are unitarily equivalent to the Schrödinger one, this seems perfectly reasonable.

It should, however, be remarked (Isham 1975) that imposing a quantum analogue of the classical condition $\det(g_{\bar{\mu}\bar{\nu}}) > 0$ could cause difficulties. It is not clear that $\hat{g}$ and $\hat{p}$ could both be self-adjoint subject to such a condition. This is an important problem but, I feel, unlikely to affect these conclusions regarding the constraint orderings. I shall not discuss it further here.

Theorem 1. If $\hat{\mathscr{H}}_{\mu}$ is an ordering of a canonically quantised translation constraint that is 'normal', then $\hat{\mathscr{H}}_{\mu}$ is symmetric in non-commuting factors.

Proof. Normality

$$\Rightarrow [\hat{\mathscr{H}}_\mu, \hat{\mathscr{H}}_\nu^+] \approx 0 \tag{20}$$

$$\Rightarrow [\hat{\mathscr{H}}_\mu, \hat{\mathscr{H}}_\nu^+ - \hat{\mathscr{H}}_\nu] \approx 0 \tag{21}$$

$$\Rightarrow [\hat{\mathscr{H}}_\mu, \hat{\mathscr{F}}_\nu] \approx 0 \tag{22}$$

$$\Rightarrow \{\mathscr{H}_\mu, \mathscr{F}_\nu\} \approx 0 \tag{23}$$

$$\Rightarrow \{\xi^\mu \mathscr{H}_\mu, \mathscr{F}_\nu\} \approx 0 \tag{24}$$

$$\Rightarrow \mathscr{L}_\xi \mathscr{F} \approx 0 \tag{25}$$

where we have defined $\hat{\mathscr{F}}_\mu := \hat{\mathscr{H}}_\mu^+ - \hat{\mathscr{H}}_\mu$ which arises from the commutation of factors in $\hat{\mathscr{H}}_\mu$. ξ^μ is an arbitrary vector flow on space–time.

Because the $\mathscr{H}$ are constrained to vanish, it is permissible to draw the ξ inside the Poisson bracket.

In going from (22) to (23) we have employed the elementary lemma

$$[\hat{\mathscr{A}}, \hat{\mathscr{B}}] = 0 \tag{26}$$

$$\Rightarrow \{\mathscr{A}, \mathscr{B}\} = 0 \tag{27}$$

the converse of which is, of course, not valid.

The symbol $\mathscr{L}_\xi$ denotes the Lie derivative with respect to the vector flow ξ. This, and its relation to equation (24), is explained in, for example, Fischer and Marsden (1972).

Thus, if $\mathscr{F} \neq 0$, it is seen that it must be a function of the spatial metric and its first derivatives whose Lie derivative with respect to an arbitrary vector flow is weakly zero.

It remains to show that $\mathscr{F} \neq 0 \Leftarrow \hat{\mathscr{F}} \neq 0$. This follows quite easily from the observation that all the terms in $\hat{\mathscr{F}}$ will occur with the same sign because $\hat{\mathscr{H}}^+$ can be made to coincide with $\hat{\mathscr{H}}$ by commuting $\hat{p}$ to the left and $\hat{g}$ to the right (say).

If $\hat{\mathscr{F}} \neq 0$ the space–time would have to satisfy the strong homogeneity condition (25). Hence, for a generic space–time, the theorem is proven.

In fact, for the ADM coordinates, a more mundane proof can be supplied. Consider the Hamiltonian constraint $\mathscr{H}_\perp$. We have in general

$$\hat{\mathscr{H}}_\perp - \hat{\mathscr{H}}_\perp^+ = \mathrm{i}\alpha g^{-1/2} g_{\bar{\mu}\bar{\nu}} p^{\bar{\mu}\bar{\nu}} \qquad \alpha \in \mathbb{R}. \tag{28}$$

Now it can be checked that a necessary condition for normality is

$$-\tfrac{3}{2}\alpha g^{-1} p^{\bar{\mu}\bar{\nu}} p^{\bar{\sigma}\bar{\tau}} (g_{\bar{\mu}\bar{\sigma}} g_{\bar{\nu}\bar{\tau}} - \tfrac{1}{6} g_{\bar{\mu}\bar{\nu}} g_{\bar{\sigma}\bar{\tau}}) = 0. \tag{29}$$

That is $\alpha = 0$. Hence

$$\hat{\mathscr{H}}_\perp - \hat{\mathscr{H}}_\perp^+ = 0. \tag{30}$$

I have included the more elaborate proof because it is rather instructive and is valid for any canonical coordinates of geometrodynamics.

6. Conclusions

I think it is fair to conclude that Komar's 'solution' is invalid. Further, due to the theorem of § 5, no such solution using non-symmetric orderings can yield a consistent theory.

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References

Almond D 1980 (*Queen Mary College Preprint* QMC-80-17) *Z. Phys.* to appear

Anderson J L 1963 in *Proc. 1st Eastern Theor. Phys. Conf.* ed M E Rose (New York: Gordon and Breach) pp 289

Arnowitt R, Deser S and Misner C W 1962 in *Gravitation, an Introduction to Current Research* ed L Witten (New York: Wiley) pp 227–65

Dirac P A M 1962 *Lectures on Quantum Field Theory, Yeshiva Lecture Notes* vol 2

Emch G 1972 *Algebraic Methods in Statistical Mechanics and Quantum Field Theory* (New York: Wiley)

Fischer A E and Marsden J E 1972 *J. Math. Phys.* **13** 436

Isham C 1975 in *Quantum Gravity, an Oxford Symposium* (Oxford: Oxford University Press)

Komar A 1979a *Phys. Rev.* D **19** 2908

—— 1979b *Phys. Rev.* D **20** 830

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